

ANSWER TO THE QUESTIONS OF YANYAN LI AND LUC NGUYEN IN ARXIV:1302.1603

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ABSTRACT. In this note we answer the two questions raised by Y.Y Li and L. Nguyen in their note [LN2] below.

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In the note [LN2], Y.Y Li and L. Nguyen raised two questions.

Q1. Whether the maximal radial function is super-harmonic.

Q2. A proof of the property $h \rightarrow 0$ as $x \rightarrow 0$ for bounded h , where

$$h(x) = w(x) - 2 \log |x|.$$

Answer to Q1: Given a lower semi-continuous function v in $B_R(x_0)$, the *maximal radial function* of v is defined by

$$\tilde{v}(x) = \inf \{v(y) : y \in \partial B_r(x_0), r = d(x, x_0)\},$$

where $B_r(x_0)$ is the geodesic ball of radius r centered at x_0 . For any $r \in (0, R)$, there is a point $x_r \in \partial B_r(x_0)$ such that $\tilde{v}(x_r) = v(x_r)$.

In page 2445, line -9, the paper [TW] contains the statement “*If v is superharmonic, then \tilde{v} is also superharmonic.*” This statement should be changed to

“*If v is superharmonic, then \tilde{v} is also superharmonic with respect to a rotationally symmetric linear operator in $B_r(x_0)$. At any point $x \in B_R(x_0)$, the coefficients of the operator are equal to those of the Laplacian at x_r . Note that by the exponential map, the Laplacian operator on a manifold in local coordinates is a linear elliptic operator with variable coefficients.*”

In [TW], we used a $W^{1,p}$ estimate for super-solutions. This estimate holds for *any* linear elliptic equations. We would like to thank Y.Y. Li and L. Nguyen for pointing out this inaccuracy in our paper.

Answer to Q2: This question was already answered in my email of November 14, 2012 to Y.Y. Li, which was included at the beginning of Section 4 in [W]. “*with the convergence in $W^{1,p}$, if the function h (h is the function in your note) is locally*

uniformly bounded, then the interior gradient estimate or the Harnack inequality (for locally bounded solutions) implies the convergence is locally uniform”.

I think if one can understand the proof of $|h| \leq C$ in page 2456, then one should see immediately $h(x) \rightarrow 0$ as $x \rightarrow 0$, by repeating the proof in page 2456 and using the interior gradient estimate. Let me give the details here.

For any sequence $x_m \rightarrow 0$, as in [TW] one makes the rescaling: $x \rightarrow x/r_m$ (with $r_m = |x_m|$) such that $\text{dist}(0, x_m) = 1$. Denote $A_r = \{x \mid 1-r < \text{dist}(0, x) < 1+r\}$ the annulus. We have shown in Lemma 3.4 [TW] that

$$(i) \quad \int_{A_{7/8}} |h| \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

From the proof of Theorem 1.3 (page 2456),

$$(ii) \quad |h| \leq C \quad \text{in } A_{3/4},$$

uniformly in m . By the interior gradient estimate, we have

$$(iii) \quad |Dh| \leq C \quad \text{in } A_{1/2},$$

uniformly in m . From (i), (ii), and (iii), we conclude that $h \rightarrow 0$ in $A_{1/4}$, uniformly. Scaling back, we obtain $h(x) \rightarrow 0$ as $x \rightarrow 0$.

Let me pointed out that the main body of the paper [TW] is to prove (i). From (i), one easily obtains (ii). The interior gradient estimate (iii) was proved in other papers.

Remark 1. When Y.Y. Li asked me Q2 in December 2012, I thought the answer was already given in [W] and didn’t bother to write more. I just simply said “*there is no need for further correspondence of this mathematics*”. For Q1, I am sure Y.Y. Li also knew the answer above.

Remark 2. I didn’t know that Y.Y. Li and L. Nguyen posted their note [LN2] on arXiv until Wednesday last week. I sent the above explanation to them last Friday but have not yet received their response for five days. So I assume my explanation is clear to them.

Based on the questions raised by Y.Y. Li in his emails and in his note [LN2], we need to make the following clarifications for the paper [TW].

- (1) (This one is copied from **Answer to Q1** above).

In page 2445, line -9, the statement “*If v is superharmonic, then \tilde{v} is also superharmonic.*” This statement should be changed to

“If v is superharmonic, then \tilde{v} is also superharmonic with respect to a rotationally symmetric linear operator in $B_r(x_0)$. At any point $x \in B_R(x_0)$, the coefficients of the operator are equal to those of the Laplacian at x_r . Note that by the exponential map, the Laplacian operator on a manifold in local coordinates is a linear elliptic operator with variable coefficients.

Accordingly, Line 1, page 2454, the sentence “*Noticing that \tilde{v}_j is superharmonic with respect to the conformal Laplace operator (1.16),*” should be changed to “*Noticing that \tilde{v}_j is superharmonic with respect to a rotationally symmetric linear elliptic operator,*”

- (2) (This one is copied from the note [W] below).

After formula (3.29) in page 2455, add

“*where $h(x) := w(x) - 2 \log |x| = o(1)$ is in the sense*

$$\lim_{r \rightarrow 0} r^{-n} \int_{\{r < |x| < 2r\}} |h(x)| dx = 0,”$$

- (3) (This is from **Answer to Q2** above).

In page 2456, line 10 after “*This is a contradiction*” add the new paragraph

“This argument also implies that $h(x) \rightarrow 0$ as $x \rightarrow 0$. Indeed, $\forall x_m \rightarrow 0$, make the above rescaling and denote $A_r = \{x \mid 1 - r < \text{dist}(0, x) < 1 + r\}$. By Lemma 3.4, we have $\int_{A_{7/8}} |h| \rightarrow 0$ as $m \rightarrow \infty$. The above paragraph tells that $|h| \leq C$. By the interior gradient estimate, $|Dh| \leq C$ in $A_{1/2}$. Hence $h \rightarrow 0$ in $A_{1/4}$ uniformly as $m \rightarrow \infty$. Scaling back, we obtain $h(x) \rightarrow 0$ as $x \rightarrow 0$.”

Acknowledgment. I would like to thank Y.Y. Li and L. Nguyen for giving me the opportunity to make the above clarifications for the paper [TW]. I am sorry that some parts of the paper was not well written and have caused difficulties for some readers to understand the proof.

REFERENCES

- [LN1] Y.Y. Li and L. Nguyen, *A compactness theorem for a fully nonlinear Yamabe problem under a lower Ricci curvature bound*, arXiv:1212.0460, Dec. 3, 2012.
- [LN2] Y.Y. Li and L. Nguyen, *Response to an article of Xu-Jia Wang*, arXiv:1302.1603, Feb. 6, 2013.
- [TW] N. Trudinger and X.-J. Wang, *The intermediate case of the Yamabe problem for higher order curvatures*, IMRN, Vol 2010, no.13, pp 2437-2458.
- [W] X.-J. Wang, *Response to a question of Yanyan Li and Luc Nguyen in their paper “A compactness theorem for a fully nonlinear Yamabe problem under a lower Ricci curvature bound”*, arXiv:1212.0460, arXiv:1212.3130, Dec. 13, 2012.

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